Situation 28: Adding Radicals

Center for Proficiency in Teaching Mathematics
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Prompt

A mathematics teacher, Mr. Fernandez, is bothered by his ninth grade algebra students' responses to a recent quiz on radicals, specifically a question in which the students added root 2 and root 3 and got root 5.

Commentary

There are many ways to think about the operations regarding radicals, and this situation will attempt to show several ways of thinking about adding radicals. Focus 1 will use a geometric approach to show how to help students develop a better sense of radicals as numbers. The reason for Focus 1—roots of whole numbers—in response to this prompt is primarily because this approach offers students the possibility of building up their knowledge of radicals as lengths (of sides of squares) so as to consider what is possible and not possible in combining and operating with these relatively "new" (for ninth graders) mathematical objects.

Focus 2 will use algebra to prove that the statement $\sqrt{2} + \sqrt{3} = \sqrt{5}$ is false. This is also related to Focus 3 in which the properties of functions are addressed, specifically when f(a + b) = f(a) + f(b). Focus 2 and Focus 3 are included because generating the notion of radicals as mathematical objects in relation to other more familiar mathematical objects like whole numbers and fractions, is crucial to constructing radicals as numbers, let alone combining and operating with them. Part of the reason that students will add

$$\sqrt{2} + \sqrt{3}$$

and get

$$\sqrt{5}$$

has to do with their familiarity with whole numbers as mathematical objects and their familiarity with combining and operating with whole numbers. Thus, a major issue to address in this situation is why those intuitions (built up over years!) do not hold. Note that students hardly have the same amount of time to build up intuitions about combining and operating with radicals. So Focus 2 and Focus 3 continue the development of this quantitative approach toward that end.

Mathematical Foci[1]

Mathematical Focus 1—roots of whole numbers

Mr. Fernandez can approach work on radicals by using a quantitative approach. That is, he can focus primarily on radicals as lengths, and then activity with radicals becomes geometrical problem solving at first, with only some numeric or algebraic calculation. Use of a tool like Geometer's Sketchpad (Jackiw, 2001) will be helpful in this regard, although some of what's mentioned below can occur without GSP.

Here's the first question the teacher can pose to students: Starting with a square of area 1 square unit (see Figure 1), can you make a square of area 2 square units?

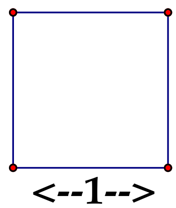

Figure 1

This problem can be solved in multiple ways, see Figure 2 below.

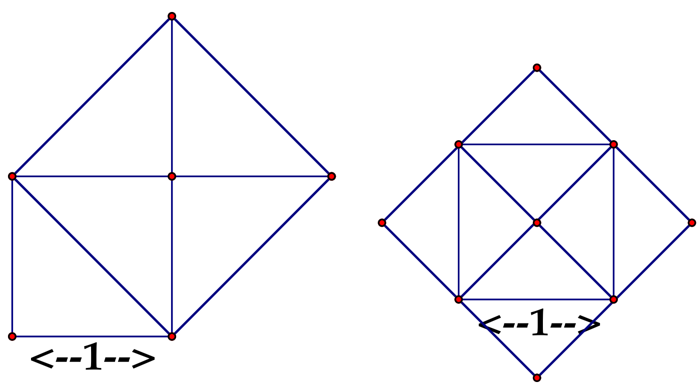

Figure 2

If a square with area 1 square unit has side root 1, which is 1, then a square with area 2 has side root 2. Some investigation here could occur with whether or not root 2 is "like the kind of numbers that we know about," i.e., whole numbers or fractions. (Most students will agree it cannot be a whole number because they can't think of a whole number that, when multiplied by itself, is 2).

Can students make squares of other areas using some of the techniques they have tried so far in making the square of area 2 (e.g., drawing diagonals and using the isosceles right triangles that

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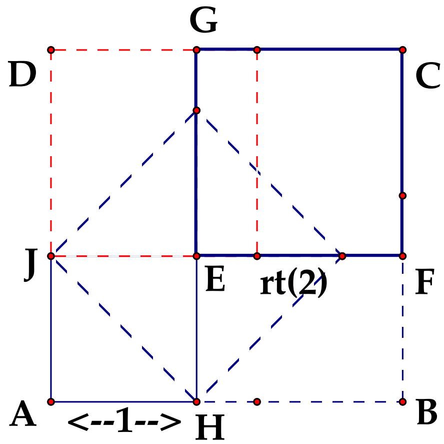
[1] The following discussion is adapted from Leslie P. Steffe's course, EMAT 7080, taught at the University of Georgia.

are formed, circumscribing squares, etc.)? What about a square of area 3 or of area 5? How can these squares be made? See Figure 3 below and **page 1** of the GSP sketch VIG C 092905 **radicals.gsp**).



Note: to move the square of area 2 so that it touches the square with area 1 at one point as shown, the blue dashed square was rotated 45 degrees to make the red dashed square and then translated 1 unit to the right. Together, the areas of AHEJ and EFCG have area 3 square units.

Figure 3

This sketch shows one way to "combine" a square of area 1 (AHEJ) and a square of area 2 (EFCG). The teacher can ask students to determine how they can use the diagram to produce a square of area 3. Note that using this diagram to make a square of area 3 is an example of the Pythagorean Theorem.

The biggest square, ABCD, consists of an area of 1 square unit, an area of 2 square units, and two rectangles that are each 1 unit by root 2 units. When the areas of the two rectangles are cut apart into 4 right triangular areas (with legs 1 and root 2 units) and separated, the remaining area in ABCD should also have area 1 + 2 = 3 (see Figure 5).

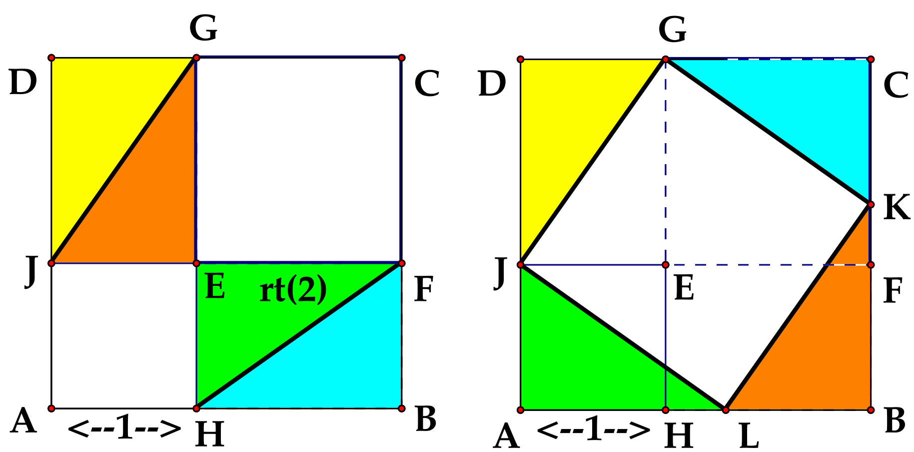


Figure 4

It remains to justify that this remaining area (GKLJ in Figure 4) is a square, which can be done using the 4 congruent right triangles (i.e., GKLJ is at least a rhombus and then it must have at least one right angle because, for example, angles AJL and DJG are complementary). So the sides of square GKLJ can be said to be

$$\sqrt{3}$$

Then the teacher can ask students to use what they know to make a square with area 5. Note that here the Pythagorean Theorem is used directly as a theorem about areas to make these area relationships and resulting lengths $\sqrt{5}$.

Students can be challenged to locate all of these lengths on a number line in order to "fill in" the number line and to develop a sense of the relationship of these radicals to numbers like the whole numbers and fractions, which they already know something about (see **pages 2, 3, and 4** of the GSP sketch VIG C 092905 **radicals.gsp**).

By using these constructions, students can then determine the relative sizes of each radical expression to help them determine the reasonableness of the equation $\sqrt{2} + \sqrt{3} = \sqrt{5}$.

One of the underlying issues in the situation is why intuitions about addition, based on addition of whole numbers, do *not* hold with fractions or radicals, while intuitions about multiplication (again based on multiplication of whole numbers) *do*. (Note that one reason for this issue is that students assume that "square rooting" distributes across addition, i.e., the erroneous concept that

$$\sqrt{2+3} = \sqrt{2} + \sqrt{3}$$

Although this kind of conception can be refuted using counterexamples, this will not be included so that this discussion will develop a quantitative approach toward this issue.) So a major reason for taking a quantitative approach to work with radicals as outlined above is to investigate this issue.

After radicals have been constructed as the lengths of sides of squares, questions can arise about how to combine these lengths. Lengths can certainly be added by joining them contiguously, but, since root 2 and root 3 cannot be written as whole numbers or fractions, we have no way to know whether we can notate their combination with a single graphic item, the way we can combine 2.5 and 3.75 into 6.25. So (at least for the moment),

$$\sqrt{2} + \sqrt{3}$$

is exactly that,

$$\sqrt{2} + \sqrt{3}$$

Furthermore, using lengths it is possible to develop intuitions about

$$\sqrt{2} + \sqrt{3}$$

NOT being equal to

$$\sqrt{5}$$

(see page 5 of VIG C 092905 **radicals.gsp**).

Mathematical Focus 2

To determine if the equation $\sqrt{2} + \sqrt{3} = \sqrt{2+3}$ is a true statement, one can use algebraic techniques to simply each side of the equation. To begin, we can square each side of the equation. When the left-hand side is squared, one must use the distribution property as well as the commutative and associative properties of addition to simply.

$$\begin{aligned}
&\left(\sqrt{2}+\sqrt{3}\right)^2 = \\
&\left(\sqrt{2}+\sqrt{3}\right)\left(\sqrt{2}+\sqrt{3}\right) = \\
&\left(\sqrt{2}+\sqrt{3}\right)\sqrt{2}+\left(\sqrt{2}+\sqrt{3}\right)\sqrt{3} = \\
&\left(\sqrt{2}\cdot\sqrt{2}+\sqrt{3}\cdot\sqrt{2}\right)+\left(\sqrt{2}\cdot\sqrt{3}+\sqrt{3}\cdot\sqrt{3}\right) = \\
&\left(\sqrt{4}+\sqrt{6}\right)+\left(\sqrt{6}+\sqrt{4}\right) = \\
&2+\sqrt{6}+\sqrt{6}+2 = \\
&4+2\sqrt{6}
\end{aligned}$$

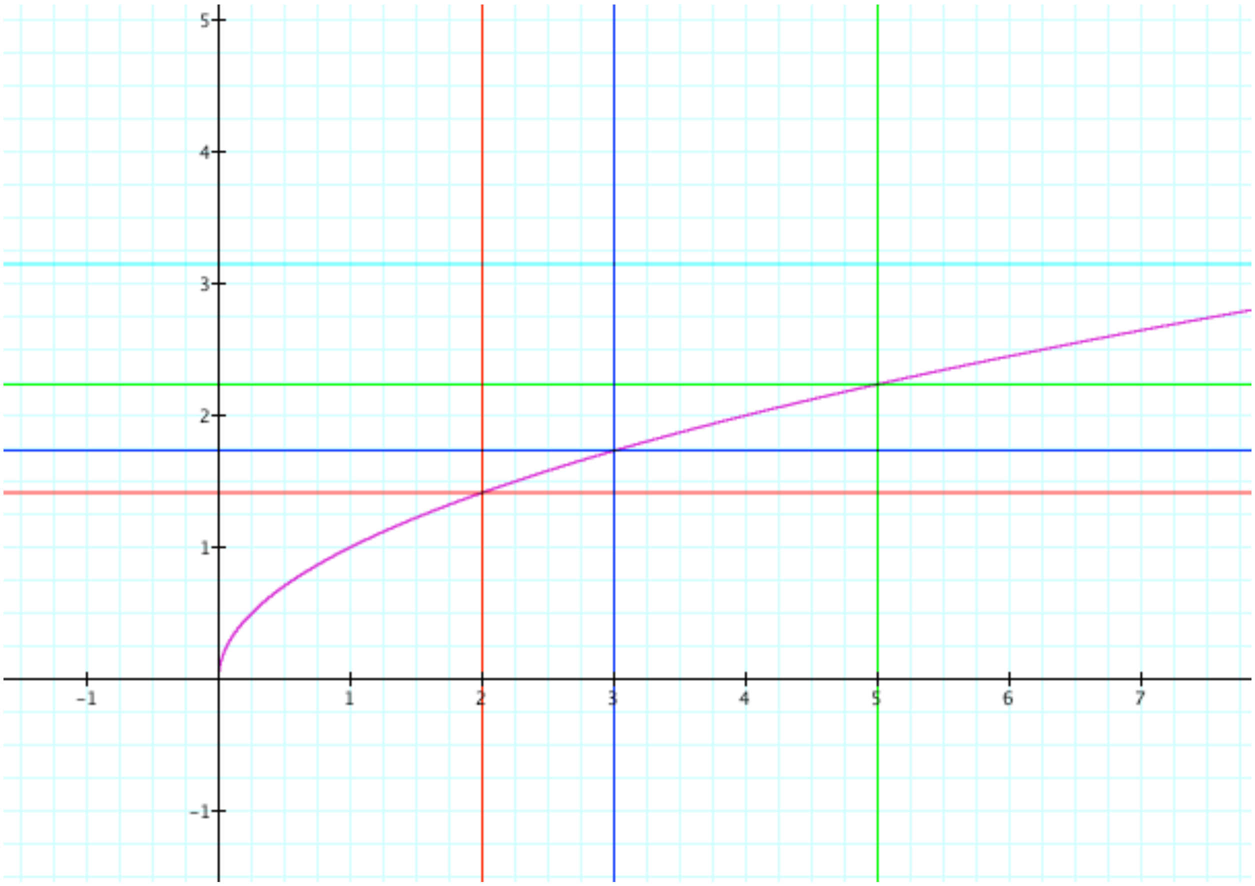
When the right-hand side is squared, however, it is immediately apparent that the sentence cannot be true. Namely,

$$\left(\sqrt{2+3}\right)^2 = 2+3 = 5.$$

Mathematical Focus 3

Another way to approach this problem is to consider the function $f(x)=\sqrt{x}$ and to think about when, if ever, $f(a+b)=f(a)+f(b)$.

Looking at the graph of $f(x)=\sqrt{x}$ will be one way to help students develop an understanding of the relative size of $f(2+3)$ and $f(2)+f(3)$. Below is a graph of the function $f(x)=\sqrt{x}$. The horizontal lines corresponding to $f(2+3)$, $f(2)$, and $f(3)$ (as well as $f(2)+f(3)$ in blue) are also shown. Using this picture, it should be clear that the square root function does not have the property that $f(a+b)=f(a)+f(b)$.



So what functions would have this property? As it turns out, a function for which this property holds is called a homomorphism. Group and ring homomorphisms are topics that are typically covered in undergraduate Algebra classes.

(What else should I say about this? Is this even worth mentioning?)

References

Jackiw, N. (2001). The Geometer's Sketchpad (Version 4.4) [Computer software]. Emeryville, CA: Key Curriculum Press.